Limits on the Network Sensitivity Function for Multi-Agent Systems on a Graph

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Abstract

This report explores the tradeoffs and limits of performance in feedback control of interconnected multi-agent systems, focused on the network sensitivity functions. We consider the interaction topology described by a directed graph and we prove that the sensitivity transfer functions between every pair of agents, arbitrarily connected, can be derived using a version of the Mason's Direct Rule. Explicit forms for special types of graphs are presented. An analysis of the role of cycles points out that these structures influence and limit considerably the performance of the system. The more the cycles are equally distributed among the formation, the better performance the system can achieve, but they are always worse than the single agent case. We also prove the networked version of Bode's integral formula, showing that it still holds for multi-agent systems.

1 Introduction

In recent years, thanks to advances in technology, attention has been focused on the control of distributed dynamical systems. In numerous mission scenarios, the concept of a group of agents cooperating to achieve a determined goal is very attractive when compared with the solution of one single vehicle. In this class of systems, even if the agents are dynamically decoupled, they are coupled through the common task they have to achieve. When the number of agents grows, centralized control is no longer feasible and distributed control techniques become attractive. Applications of coordinated control of multiple vehicles can be found in many fields, including microsatellite clusters [1, 2], formation flying of unmanned aerial vehicles [3], automated highway systems [4] and mobile robotics [5].

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The problem of distributed control has been widely studied with tools from graph theory [6, 7, 8]. We consider agents with identical linear dynamics and we model the interconnection topology as a graph, in which the single agents are represented by a vertex, while the interaction links are the arcs.

The distributed control problem has been handled in different ways and with different tools: dissipative theory and linear matrix inequalities [9], edge agreement [10, 11], linear quadratic regulators [12], and decomposition and linear matrix inequalities [13]. In all the works mentioned above the control is applied to undirected graphs. If the graph is undirected the problem becomes easier because all the matrices associated with the graph, like the Laplacian, are symmetric. In the present paper we will consider the more general case of directed graphs.

One approach to distributed control is to use leader-follower arrangement. This approach is well studied and representative papers exploring graph-theoretic ideas in the context of a leader-follower architecture include [8] and [14]. This topology represents a particular case, where the leader has a more important role than the other agents and this may not always be desirable. In our work we explore a broader set of architectures, including leader-follower as a special case.

Jin in [15] proposed a double-graph control strategy in order to improve stability of the interconnected system and to relate stability conditions and performance of disturbance resistance on the connectivity of the graph. Jin considered arbitrary directed graphs and he supposed to have two controllers: one for global objective and the other for local interaction. In his work he distinguished between weakly and strongly connected graphs, but there is a lack on the more general analysis of the interaction topology. Moreover the global objective was supposed to be known instantaneously among the formation.

The importance of cycles in distributed control has already been pointed out in several past works: Zelazo et al. [10, 11] investigated the role of cycles and trees in the edge Laplacian for the edge agreement problem, while Fax and Murray [16] suggested a relation between the presence of cycles and the stability of formation. In this paper we explicitly relate some graph substructures, such as cycles and directed paths, to system behavior and disturbance rejection.

The contribution of this paper is to show a general method to derive the transfer functions between any pair of agents, where the interconnection topology is described by arbitrary directed graphs and the leader-follower architecture is only a particular case. We start from classical control theory and we define the basic concepts of stability margins, transfer functions and loop shaping in order to deal with multi-agent systems. We then analyze mechanisms that rule the behavior of a multi-agent system and we show intrinsic limits on the controller design due to the topology.

The paper is organized as follows. In section 2 we briefly review the principal concepts of graph theory and the main stability results on formation control. Section 3 extends several classical control concepts in order to deal with multi-agent systems. The core of the paper is presented in Section 4, where we prove the formula to derive all the networked sensitivity functions for an arbitrary number of agents and topology. In Section 5 we show some networked sensitivity functions on special graphs, while in Section 6 some design considerations and limitations are proposed and the networked version of Bode's integral formula is proved. Finally, Section 7 contains three examples on different interaction topologies and the conclusions of the report are given in Section 8.

2 Preliminaries

In this section we summarize some of the key concepts and definitions from graph theory that will be used in the paper. A more detailed presentation of graph theory can be found in [17].

A *directed graph* $\mathcal{G}$ is a set of vertices or nodes V and a set of arcs $A \subset V^2$ whose elements $a = (u, v) \in A$ characterize the relation between distinct pairs of vertices $u, v \in V$. For an arc (u, v) we call u the *tail* and v the *head*. The *in(out)degree* of a vertex v is the number of arcs with v as its head (tail). A *subgraph* of a graph $\mathcal{G}$ is a graph whose vertex and arc sets are subsets of those of $\mathcal{G}$. A *directed path* in a graph is a sequence of vertices such that from each of its vertices there is an arc to the next vertex in the sequence. A directed path with no repeated vertices is called a *simple directed path*. A directed graph is called *strongly connected* if there is a directed path from each vertex in the graph to every other vertex. A directed graph is *weakly connected* if every vertex can be reached from every other but not necessary following the directions of the arcs. A *complete directed graph* is a graph where each pair of vertices has an arc connecting them. We write $|V|$ for the number of vertices in a graph.

The structure of a graph can be described by some matrices. The *adjacency matrix* $\mathcal{A}$ of a graph $\mathcal{G}$ is a square matrix of size $|V|$, defined by $\mathcal{A}_{ij} = 1$ if $(i, j) \in A$, and zero otherwise. The *normalized Laplacian matrix* $\mathcal{L}$ of a directed graph $\mathcal{G}$ is a square matrix of size $|V|$, defined by

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\dfrac{1}{d_{o_i}} & \text{if } (i, j) \in A \\ 0 & \text{otherwise,} \end{cases}$$

where d_{o_i} is the outdegree of the ith vertex. We observe that if $d_{o_i} > 0$ for all vertices in the graph, $\mathcal{L}$ has zero row sum, which implies that zero is an eigenvalue of $\mathcal{L}$. Furthermore if $\mathcal{G}$ is strongly connected, zero is a simple eigenvalue of $\mathcal{L}$ and all eigenvalues of $\mathcal{L}$ lie in a disk in the complex plane with unity radius and centered at $1 + 0\jmath$ [18].

We consider a formation of N agents with identical linear dynamics. Each dynamical element corresponds to a node of the graph and the normalized Laplacian matrix $\mathcal{L}$ is used to represent the interaction topology. Suppose each individual agent is a SISO system with a local feedback loop composed of a local controller $C(s)$ and a plant model $P(s)$. According to Fax [16], a local controller stabilizes the whole formation if and only if it simultaneously stabilizes N subsystems. Specifically, the multi-agent system is stable if and only if the net encirclement of the critical points $-\lambda_i^{-1}(\mathcal{L})$ by the Nyquist plot of $P(s)C(s)$ is zero for all nonzero $\lambda_i(\mathcal{L})$, where $\lambda_i(\mathcal{L})$ are the eigenvalues of the normalized Laplacian matrix $\mathcal{L}$ of the graph.

We will utilize some additional notation used by Fax [16]. The Kronecker product $\otimes$ between two matrices $P = [p_{ij}]$ and $Q = [q_{ij}]$ is defined as

$$P \otimes Q = [p_{ij}Q].$$

This is a block matrix with the ijth block of $p_{ij}Q$. Let I_n indicate the identity matrix of order n. To represent the matrix M repeated N times along the diagonal we write

$$\widehat{M} = I_N \otimes M.$$

Letting n be the number of configuration (output) variables of each agent that can be controlled, $\mathcal{L}_{(n)}$ is of dimension $Nn \times Nn$, i.e. $\mathcal{L}_{(n)} = \mathcal{L} \otimes I_n$.

3 Stability and performance measure

In this section we will investigate how the performance specifications for single agent control systems translate into requirements for multi-agent systems. The concepts of sensitivity functions and stability margins will be extended to reflect the interconnection topology.

We consider the multi-agent feedback system in Figure 1, where $\mathbf{r} \in R^N$ is the vector of the reference signals of each agent, $\mathbf{e} \in R^N$ are the errors between $\mathbf{r}$ and the process outputs $\mathbf{y} \in R^N$, $\mathbf{u} \in R^N$ is the control signal vector and $\mathbf{d} \in R^N$ and $\mathbf{n} \in R^N$ are the load disturbances and the measurement noises respectively.

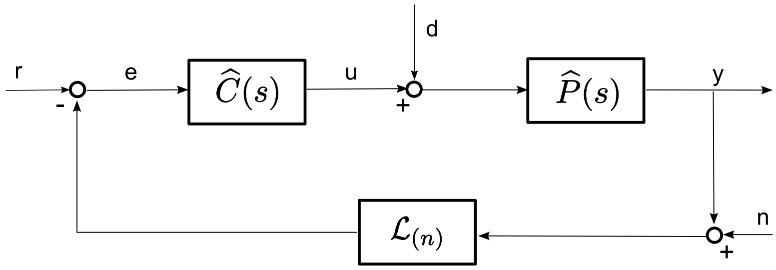


Figure 1: Block diagram of a multi-agent feedback system.

Define the *networked loop transfer function matrix* as

$$\widehat{L}(s) = \widehat{P}(s)\widehat{C}(s).$$

Throughout the report we will consider only stable systems. The relations between the inputs and the interesting signals of the system are given by the following transfer function matrices, which can be recognized to be the networked version of the single agent equivalents. We define the *networked sensitivity function matrix* $\widetilde{S}(s)$ as

$$\widetilde{S}(s) = \left(I + \mathcal{L}_{(n)}\widehat{P}(s)\widehat{C}(s)\right)^{-1},$$

the *networked complementary sensitivity function matrix* $\widetilde{T}(s)$ as

$$\widetilde{T}(s) = \left(I + \mathcal{L}_{(n)}\widehat{P}(s)\widehat{C}(s)\right)^{-1}\widehat{P}(s)\widehat{C}(s),$$

the *networked load sensitivity function matrix* $\widehat{P}(s)\widetilde{S}(s)$ as

$$\widehat{P}(s)\widetilde{S}(s) = \left(I + \mathcal{L}_{(n)}\widehat{P}(s)\widehat{C}(s)\right)^{-1}\widehat{P}(s),$$

and the *networked noise sensitivity function matrix* $\mathcal{L}_{(n)}\widehat{C}(s)\widetilde{S}(s)$ as

$$\mathcal{L}_{(n)}\widehat{C}(s)\widetilde{S}(s) = \left(I + \mathcal{L}_{(n)}\widehat{P}(s)\widehat{C}(s)\right)^{-1}\mathcal{L}_{(n)}\widehat{C}(s).$$

From now on, without loss of generality, we will consider $n = 1$ so that each agent has a single configuration variable (output) that is being controlled. By analogy with the single agent case, in order to guarantee stability, robustness and good performance, we want to have

$$|\widetilde{S}(j\omega)| \ll 1 \text{ for } \omega \ll \omega_c, \text{ and } |\widetilde{S}(j\omega)| \approx 1 \text{ for } \omega \gg \omega_c,$$

$$|\widetilde{T}(j\omega)| \approx 1 \text{ for } \omega \ll \omega_c, \text{ and } |\widetilde{T}(j\omega)| \ll 1 \text{ for } \omega \gg \omega_c,$$

where ω_c is the critical frequency that describes our desired bandwidth.

Since in a multi-agent control the critical point for the stability of the system is no longer the point -1, but the collection of points $-\lambda_i^{-1}(\mathcal{L})$, the well-know indicators for how near the Nyquist plot is to the critical points need to be redefined.

Define the *networked gain margin* GM_n as the minimum scaling that will cause the Nyquist curve for $L(j\omega)$ intersect one of the eigenvalues of $\mathcal{L}$, as shown in Figure 2a:

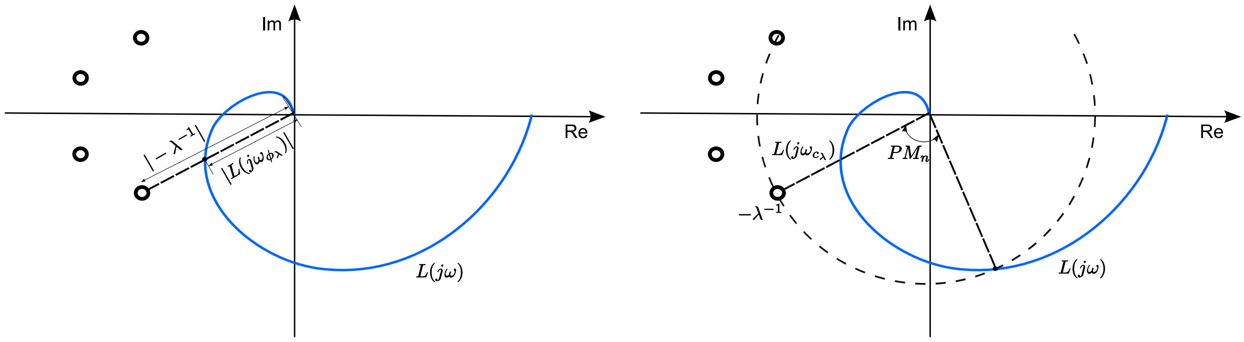
$$GM_n = \min_i \frac{1}{|\lambda_i||L(j\omega_{\phi_\lambda})|}.$$

Similarly, define the *networked phase margin* PM_n as the minimum angle between the argument of $-\lambda^{-1}$ and $L(j\omega_{c_\lambda})$, where ω_{c_λ} is the angular frequency where the Nyquist plot intersects the circle with radius $|-\lambda^{-1}|$ closest to the point $-\lambda^{-1}$, i.e. where $|L(j\omega)| = |-\lambda^{-1}|$ (Figure 2b):

$$PM_n = \min_i\{\arg(L(j\omega_{c_\lambda})) - \arg\left(-\lambda_i^{-1}\right)\}.$$

Since $\arg\left(-\lambda_i^{-1}\right) = -\arg(\lambda_i) + \pi$, the networked phase margin can also be rewritten as

$$PM_n = \min_i\{\arg(L(j\omega_{c_\lambda})) + \arg(\lambda_i) - \pi\}.$$



(a) Networked gain margin (b) Networked phase margin

Figure 2: Networked gain and phase margin.

4 Disturbance rejection for multi-agent systems

In this section we show how to derive the networked sensitivity transfer functions between any pair of agents for a given topology. We are dealing with determining transfer functions on graphs and an effective and straightforward means to achieve it is signal-flow graph theory. A signal-flow graph is a diagram that depicts the cause and effect relationship among a number of variables. The main results in this area are due to Mason [19], who derived a rule to compute the transfer function of a signal-flow graph, commonly known as *Mason's Direct Rule* [20, 21]. In the following we will build our work borrowing tools from signal-flow graph theory.

We begin by looking at some example cases to explore the different possible behaviors. If we look at the components of the networked sensitivity function matrix $|\widetilde{S}_{ij}|$ for arbitrary topology, we can observe that similar topologies have similar behavior. For example in an acyclic graph, the magnitude of all the components of $|\widetilde{S}|$ goes to zero as $|PC|$ increases (Figure 3); in a complete directed graph the magnitude of all the components of $|\widetilde{S}|$ asymptotically goes to $1/N$ as $|PC|$ increases (Figure 4); in a graph with cycles of different lengths the magnitude of the components of $\widetilde{S}$ asymptotically goes to n values to be determined (Figure 5). These few cases suggest that not only the number of agents N, but also the topology and graph substructures, like paths and cycles, deeply influence the behavior of interconnected systems. For this reason a more detailed analysis is needed.

We define the *Laplacian weight* of a simple directed path of length k from i to j, where $i = i_0,\ i_1, \dots, i_k = j$, as the product of the negative inverse of the outdegrees d_o of all the nodes in the path besides the last one:

$$\mathcal{L}w^k_{i_0 i_k} := \operatorname{sgn}(k) \prod_{t=0}^{t=k-1} \left(-\frac{1}{d_{o_{i_t}}}\right), \tag{1}$$

where $\operatorname{sgn}(k) = -1$ if k is odd, $\operatorname{sgn}(k) = +1$ if k is even. We say a path is a *degenerate path* if it is a path of length zero between a node and itself and we define its Laplacian weight as one: $\mathcal{L}w^0_{ii} = 1$. A simple cycle of length k is a closed path through k connected links that is self-avoiding (does not revisit nodes, other than the first) [22]. Since in a cycle every node can

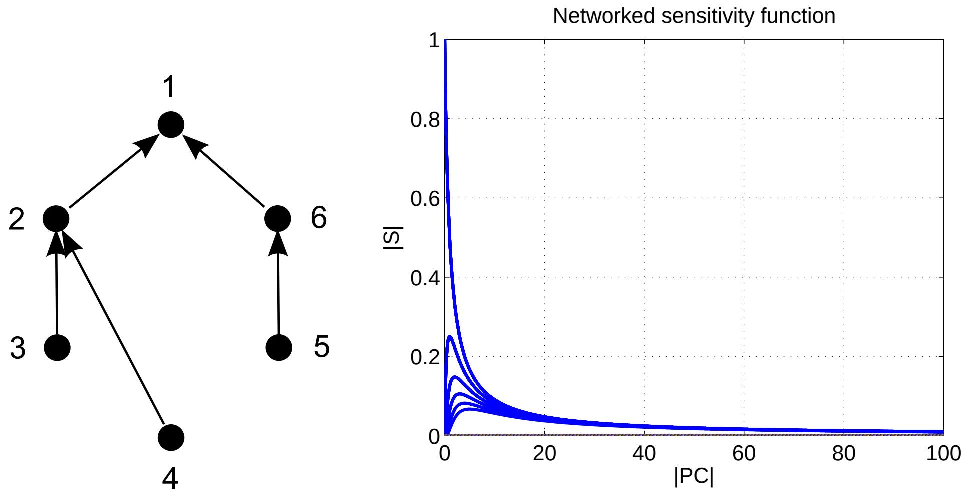


Figure 3: Networked sensitivity functions for acyclic graph, $N = 6$.

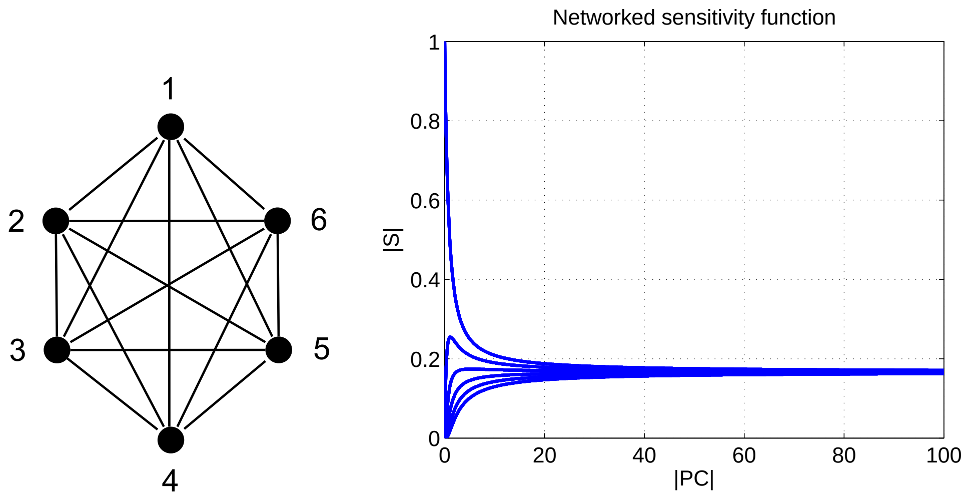


Figure 4: Networked sensitivity functions for 6 complete graph. The asymptotic value is $1/6$.

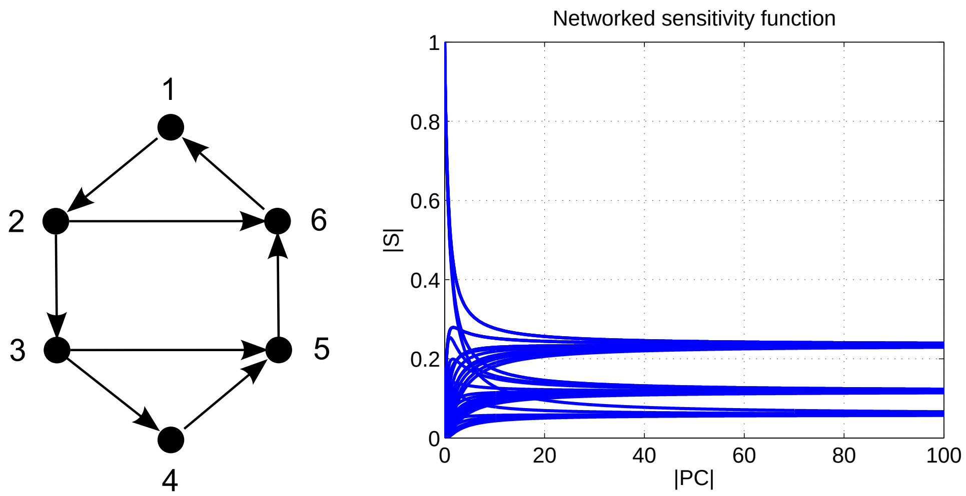


Figure 5: Networked sensitivity functions for a graph with 3-cycle, 5-cycle and 6-cycle.

be the starting and ending node, the Laplacian weight of a cycle will be indicated with o as subscript. The Laplacian weight of a cycle of length k will be

$$\mathcal{L}w_o^k := \operatorname{sgn}(k-1) \prod_{t=0}^{t=k-1} \left(-\frac{1}{d_{o_{i_t}}} \right), \ i_0 = i_k, \tag{2}$$

We define *disjoint cycles* in $\mathcal{G}$ to be a set of non-adjacent simple cycles, that is, two simple cycles that do not share any common nodes. An example is shown in Figure 6. The length

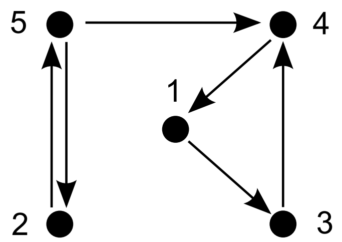
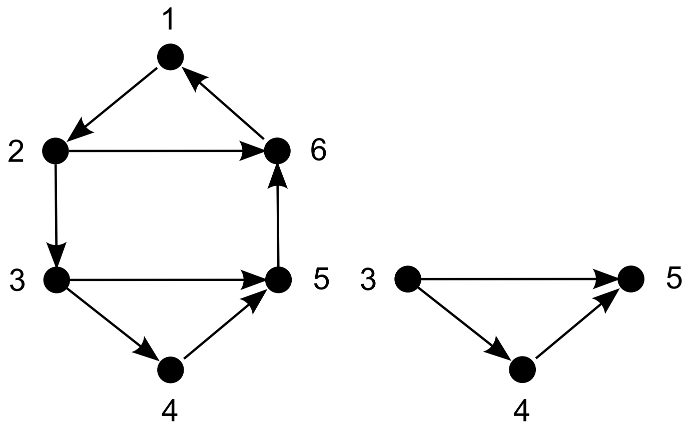


Figure 6: Example of disjoint cycles: 2-5-2 and 1-3-4-1.

of disjoint cycles is given by the sum of the lengths of the composing simple cycles, while the Laplacian weight of disjoint cycles is given by the product of the Laplacian weights of the composing simple cycles. Given two simple cycles a and b of length $k_{(a)}$ and $k_{(b)}$ and Laplacian weight of $\mathcal{L}w_{o(a)}$ and $\mathcal{L}w_{o(b)}$ respectively, the disjoint cycles composed by a and b will have the length $k_{(ab)} = k_{(a)} + k_{(b)}$ and Laplacian weight $\mathcal{L}w_{o(ab)} = \mathcal{L}w_{o(a)} \cdot \mathcal{L}w_{o(b)}$.

Define $\mathcal{G}_{ij}^k$ as the subgraph of $\mathcal{G}$ obtained from $\mathcal{G}$ by removing all the nodes and all the arcs touching the simple directed path from node i to node j of length k. An example is shown in Figure 7.



(a) Graph $\mathcal{G}$ (b) Subgraph $\mathcal{G}_{16}^2$ of the path $1-2-6$

Figure 7: Subgraph example.

Theorem 1. *The sensitivity transfer function between every pair of nodes i and j of a generic graph $\mathcal{G}$ can be derived using the following expression, which is a version of the* Mason's Direct *Rule [20, 21]:*

$$\widetilde{S}_{ij} = \frac{1}{\Delta} \sum_{paths\ p\ \in\ \mathcal{G}} T_p \Delta_p, \tag{3}$$

where:

(i) Δ is the determinant of $(I + \mathcal{L}\widehat{P}\widehat{C})$,

$$\Delta = (1 + PC)^N + \sum_{cycles\ o\ \in\ \mathcal{G}} \left(\mathcal{L}w_o^k\right)(1 + PC)^{(N-k)}(PC)^k; \tag{4}$$

(ii) T_p is the 'gain' of the p^{th} simple directed path from node i to node j of length k,

$$T_p = \left(\mathcal{L}w_{ij}^k\right)(PC)^k; \tag{5}$$

and

(iii) Δ_p is the value of Δ for the subgraph $\mathcal{G}_{ij}^k$ not touching the p^{th} simple directed path from node i to node j of length k,

$$\Delta_p = (1 + PC)^{(N-1-k)} + \sum_{cycles\ o\ \in\ \mathcal{G}_{ij}^k} \left(\mathcal{L}w_o^{\bar{k}}\right)(1 + PC)^{(N-1-k-\bar{k})}(PC)^{\bar{k}}, \tag{6}$$

and $\bar{k}$ represents the length of the cycles in $\mathcal{G}_{ij}^k$.

Proof. For a signal flow graph $\mathcal{G}$, the gain matrix M [19] is

$$M = (I - \bar{\mathcal{A}})^{-1}, \tag{7}$$

where $\bar{\mathcal{A}}$ is the weighted adjacency matrix associated with the signal flow. Suppose now instead of having $\mathcal{G}$ we have a transformed graph $\widetilde{\mathcal{G}}$ (as the example in Figure 8),with the same topology of $\mathcal{G}$ but with the weight of each arc equal to

$$w_{ij} = \frac{1}{d_{o_i}} PC,\ \forall (i,j) \in \widetilde{\mathcal{G}}$$

and self-loops in each node with weight

$$w_{ii} = -PC,\ \forall i \in \widetilde{\mathcal{G}}.$$

We take the generic case of complete directed graph. In this way the transformed weighted adjacency matrix $\widetilde{\mathcal{A}}$ for the graph $\widetilde{\mathcal{G}}$ will be:

$$\widetilde{\mathcal{A}} = \begin{bmatrix} -PC & \frac{1}{d_{o_1}}PC & \cdots & \frac{1}{d_{o_1}}PC & \frac{1}{d_{o_1}}PC \\ \frac{1}{d_{o_2}}PC & -PC & \cdots & \frac{1}{d_{o_2}}PC & \frac{1}{d_{o_2}}PC \\ \vdots & & \ddots & & \vdots \\ \frac{1}{d_{o_{N-1}}} & \frac{1}{d_{o_{N-1}}} & \cdots & -PC & \frac{1}{d_{o_{N-1}}} \\ \frac{1}{d_{o_N}} & \frac{1}{d_{o_N}} & \cdots & \frac{1}{d_{o_N}} & -PC \end{bmatrix} = -\mathcal{L}\widehat{P}\widehat{C}.$$

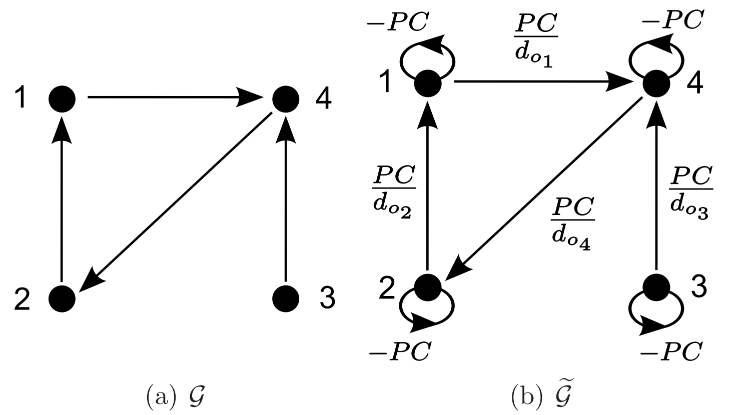


Figure 8: Example of transformation from a graph $\mathcal{G}$ to the signal flow graph $\widetilde{\mathcal{G}}$.

Applying the equation (7), we get the transformed gain matrix $\widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{G}}$:

$$\widetilde{\mathcal{M}} = (I - \widetilde{\mathcal{A}})^{-1} = (I + \mathcal{L}\widehat{P}\widehat{C})^{-1}.$$

This is exactly what we need to solve in order to compute the matrix sensitivity transfer function. Applying the Mason's direct rule to $\widetilde{\mathcal{G}}$ we obtain exactly the denominator and numerators in equations (4), (5) and (6), where the self loops generate the loop difference $(1 + PC)$. ■

In classical control theory, in order to attenuate the disturbances entering the system, the gain of S is reduced at low frequencies and consequently the gain of the open loop transfer function is large at those frequencies. Therefore it is interesting to study the asymptotic behavior of the networked sensitivity functions for $|PC| \to \infty$.

The denominator in equation (3) is the determinant of $(I + \mathcal{L}\widehat{P}\widehat{C})$, so it is a polynomial of Nth order in PC. It depends only on the cycles in $\mathcal{G}$ and it is the same for all the $\widetilde{S}_{ij}$ and $\widetilde{S}_{ii}$.

Proposition 2. *Given a graph $\mathcal{G}$, the determinant of the normalized Laplacian matrix $\mathcal{L}$ is*

$$\det(\mathcal{L}) = 1 + \sum_{cycles\ o\ \in\ \mathcal{G}} \left(\mathcal{L}w_o^k\right). \tag{8}$$

Proof. We will follow the proof of Theorem 1. Consider the transformed graph $\widetilde{\mathcal{G}}$ with the same topology described by $\mathcal{L}$ but the weight of each arc equal to

$$w_{ij} = \frac{1}{d_{o_i}},\ \forall (i,j) \in \widetilde{\mathcal{G}}$$

and no self-loops ($w_{ii} = 0$). The transformed weighted adjacency matrix $\widetilde{\mathcal{A}}$ for the graph $\widetilde{\mathcal{G}}$ will be $\widetilde{\mathcal{A}} = I - \mathcal{L}$ and the transformed gain matrix $\widetilde{\mathcal{M}} = (I - I + \mathcal{L})^{-1} = (\mathcal{L})^{-1}$. The denominator of the gain matrix is the determinant of $\mathcal{L}$. ■

Theorem 3. *If every vertex in $\mathcal{G}$ has outdegree greater than zero, the coefficient of $(PC)^N$ in the complete polynomial expression of the denominator is always zero:*

$$1 + \sum_{\textit{cycles } o \, \in \, \mathcal{G}} \left(\mathcal{L}w_o^k\right) = 0. \tag{9}$$

Proof. The coefficient of $(PC)^N$ can be computed from equation (4) and gives the left hand side of equation (9). This term is the determinant of the normalized Laplacian matrix $\mathcal{L}$ by Proposition 2. For graphs with $d_{o_i} > 0$ for every node we already know that $\mathcal{L}$ has a zero eigenvalue and so $\det(\mathcal{L}) = 0$. ■

Therefore for weakly connected graphs with outdegree of every node greater than zero, the polynomial in the denominator is order $N-1$ in PC. The asymptotic value as $|PC| \to \infty$ depends on the coefficient of $(PC)^{(N-1)}$ and it is easy to show it is given by

$$N + \sum_{\text{cycles } o \, \in \, \mathcal{G}} \left(\mathcal{L}w_o^k\right)(N-k). \tag{10}$$

If a graph has at least one node with outdegree equal to zero, the Laplacian matrix looses its property of zero row sum and $\det(\mathcal{L}) \neq 0$. Graphs of this type will have a polynomial in the denominator of Nth order.

The numerator of $\widetilde{S}_{ij}$ is given by equations (5) and (6):

$$\begin{aligned} \sum_{\text{paths } ij \, \in \, \mathcal{G}} & \left(\left(\mathcal{L}w_{ij}^k\right)(PC)^k\right) \cdot \\ & \left((1+PC)^{(N-1-k)} + \sum_{\text{cycles } o \, \in \, \mathcal{G}_{ij}^k} \left(\mathcal{L}w_o^{\overline{k}}\right)(1+PC)^{(N-1-k-\overline{k})}(PC)^{\overline{k}}\right). \end{aligned} \tag{11}$$

It is an element of the adjugate matrix (the transpose of the cofactors matrix) of $(I + \mathcal{L}\widehat{P}\widehat{C})$ and it is a polynomial of order $N-1$ in PC. The coefficients depend on all the simple directed paths from node i to node j and on the cycles of the subgraphs $\mathcal{G}_{ij}^k$. The value of the coefficient of $(PC)^{(N-1)}$ in the complete polynomial expression of the numerator for $i \neq j$ is given by

$$\sum_{\text{paths } ij \, \in \, \mathcal{G}} \left(\left(\mathcal{L}w_{ij}^k\right) \cdot \left(1 + \sum_{\text{cycles } o \, \in \, \mathcal{G}_{ij}^k} \left(\mathcal{L}w_o^{\overline{k}}\right)\right)\right). \tag{12}$$

If no cycles exist in $\mathcal{G}_{ij}^k$, then $\mathcal{L}w_o^{\overline{k}} = 0$. If no path exists from node i to node j, $\widetilde{S}_{ij}$ will be always zero for every $|PC|$ value.

If $i = j$ we have a degenerate path and $\mathcal{L}w_{ii}^0 = 1$. The subgraph $\mathcal{G}_{ij}^k$ is obtained by removing the ith node and all the arcs with head or tail in i and it will be indicated by $\mathcal{G}_i$. We have only to look at the Laplacian weights $\mathcal{L}w_o$ of all the simple cycles or disjoint cycles in $\mathcal{G}_i$. Equations (5) and (6) for $i = j$ simplify to

$$(1+PC)^{(N-1)} + \sum_{\text{cycles } o \, \in \, \mathcal{G}_i} \left(\mathcal{L}w_o^{\overline{k}}\right)(1+PC)^{(N-1-\overline{k})}(PC)^{\overline{k}}, \tag{13}$$

and the value of the coefficient of $(PC)^{(N-1)}$ in the complete polynomial expression of the numerator for $i = j$ becomes

$$1 + \sum_{\text{cycles } o \,\in\, \mathcal{G}_i} \left(\mathcal{L} w_o^{\bar{k}}\right). \tag{14}$$

The denominator of the multi-agent sensitivity functions has an important meaning. It is the determinant of $(I + \mathcal{L}\widehat{P}\widehat{C})$ and it can be expressed using the Schur transformation $\mathcal{L} = TUT^{-1}$, where T is a unitary matrix and U is an upper triangular matrix with the eigenvalues λ_i of $\mathcal{L}$ along the diagonal [23]:

$$\begin{aligned}
\det\left[I + \mathcal{L}\widehat{P}\widehat{C}\right] &= \det\left[I + TUT^{-1}\widehat{P}\widehat{C}\right] \\
&= \det\left[T\left(I + U\widehat{P}\widehat{C}\right)T^{-1}\right] \\
&= \det\left[I + U\widehat{P}\widehat{C}\right] \\
&= \prod_{i=1}^{N} \det\left[1 + \lambda_i PC\right] = \prod_{i=1}^{N} \left(1 + \lambda_i PC\right).
\end{aligned} \tag{15}$$

From equation (15) it is evident that the eigenvalues of $\mathcal{L}$ are involved in the denominator. If a graph has all vertices with outdegree greater than zero, $\mathcal{L}$ has a zero eigenvalue and $\lambda_1 = 0$ is responsible for the $N - 1$ polynomial's degree.

The numerator of $\widetilde{S}_{ij}$ is the cofactor of the element ij of the matrix $(I + \mathcal{L}\widehat{P}\widehat{C})$. From matrix algebra we know that the cofactor of an element ij, which we call Δ_{ij}, is obtained by the determinant of the minor for entry ij, removing row i and column j. Therefore in the numerator will be involved the eigenvalues of the first minors of $\mathcal{L}$, which are not known a priori.

From Theorem 3 we can assert that if every node has outdegree greater than zero, both $\widetilde{S}_{ij}$ and $\widetilde{S}_{ii}$ are proper functions in terms of the open loop transfer function. If at least one node has $d_o = 0$, $\widetilde{S}_{ij}$ and $\widetilde{S}_{ii}$ are strictly proper functions.

5 Sensitivity functions on special graphs

To better understand the sensitivity functions described in the last section, we consider some simple examples. In all the following examples no self-loops will be considered.

5.1 Complete directed graph

Theorem 4. *For a complete directed graph with N nodes, the networked sensitivity function matrix $\widetilde{S}$ is composed of only two different sensitivity transfer functions, $\widetilde{S}_{ii}$ and $\widetilde{S}_{ij}$, where:*

$$\widetilde{S}_{ii} = \frac{PC + (N-1)}{N \cdot PC + (N-1)} \tag{16}$$

and

$$\widetilde{S}_{ij} = \frac{PC}{N \cdot PC + (N-1)}. \tag{17}$$

The asymptotic value for both when $|PC| \to \infty$ is equal to $1/N$.

Proof. The diagonal sensitivity functions can be expressed as $\widetilde{S}_{ii} = \Delta_{ii}/\Delta$, where Δ_{ii} is the symmetric first minor of $(I + \mathcal{L}\widehat{P}\widehat{C})$ and $\Delta = \det\left[I + \mathcal{L}\widehat{P}\widehat{C}\right]$. In a complete directed graph one eigenvalue is equal to zero, while all the others are $\lambda = N/(N-1)$ [18]. From equation (15) we have

$$\Delta = \left(1 + \frac{N}{N-1}PC\right)^{(N-1)}. \tag{18}$$

The same can be done for Δ_{ii}, but first we need to know the eigenvalues of the symmetric first minors of $\mathcal{L}$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $\mathcal{L}$, with $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_n = N/(N-1)$. Let $\mu_1 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of the symmetric first minors of $\mathcal{L}$. We need two important linear algebra properties:

1. the interlacing eigenvalues theorem
2. the trace of a square matrix A is the sum of the eigenvalues of A.

From the first we have $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \mu_{n-1} \geq \lambda_n$ and therefore $\mu_i = N/(N-1)$ with $i \neq 1$. From the latter and recalling that the elements on the diagonal of $\mathcal{L}$ are all equal to one, we can find $\mu_1 = 1/(N-1)$. So we have

$$\Delta_{ii} = \left(1 + \frac{N}{N-1}PC\right)^{(N-2)} \left(1 + \frac{1}{N-1}PC\right). \tag{19}$$

Finally putting together equations (18) and (19) we can prove the equation (16):

$$\widetilde{S}_{ii} = \frac{\Delta_{ii}}{\Delta} = \frac{\left(1 + \frac{N}{N-1}PC\right)^{(N-2)} \left(1 + \frac{1}{N-1}PC\right)}{\left(1 + \frac{N}{N-1}PC\right)^{(N-1)}} = \frac{PC + (N-1)}{N \cdot PC + (N-1)}.$$

To prove the equation (17) we need to observe that the numerator of $\widetilde{S}_{ij}$ is the ijth first minor of $(I + \mathcal{L}\widehat{P}\widehat{C})$, that we will indicate with Δ_{ij}, and to recall that the determinant of a matrix can be written as the sum of its cofactors multiplied by the entries that generated them. Since the graph is fully symmetric, there is no reason why the cofactors of the extra diagonal elements should be different. So we have the cofactor expansion along a row:

$$\Delta = (1 + PC)\Delta_{ii} + (N-1)\left(-\frac{1}{N-1}PC\right)\Delta_{ij}, \tag{20}$$

where the term $(N-1)$ counts for the identical extra diagonal elements. Replacing equations (18) and (19) in the equation (20) we can find Δ_{ij}:

$$\Delta_{ij} = \frac{1}{N-1}PC\left(1 + \frac{N}{N-1}PC\right)^{(N-2)}.$$

Finally we can prove the equation (17):

$$\widetilde{S}_{ij} = \frac{\Delta_{ij}}{\Delta} = \frac{\frac{1}{N-1}PC\left(1+\frac{N}{N-1}PC\right)^{(N-2)}}{\left(1+\frac{N}{N-1}PC\right)^{(N-1)}} = \frac{PC}{N\cdot PC+(N-1)}.$$

■

5.2 Directed tree

A polytree is a graph with at most one undirected path between any two vertices. In other words, a polytree is a directed acyclic graph for which there are also no undirected cycles. In particular, for directed trees every node should have outdegree 0 or 1. A tree is a graph in which any two vertices are connected by exactly one path. In other words, any connected graph without cycles is a tree. Every directed tree is a polytree, but not every polytree is a directed tree. An example of directed tree graph is shown in Figure 3.

Theorem 5. *For a directed tree graph, the networked sensitivity function matrix $\widetilde{S}$ is independent from the number of agents N and it is given by*

$$\widetilde{S}_{ii} = \frac{1}{PC+1} \tag{21}$$

and for i and j connected by a path

$$\widetilde{S}_{ij} = \frac{(PC)^k}{(1+PC)^{(k+1)}}, \tag{22}$$

where k is the length of the only path from i to j. The asymptotic value when $|PC|\to\infty$ is always equal to zero.

Proof. First of all since there are no cycles in the graph $\mathcal{L}w_o = 0$ the numerator of $\widetilde{S}_{ij}$ reduces to

$$\sum_{\text{paths } ij\,\in\,\mathcal{G}} \left(\mathcal{L}w_{ij}^k\right)(1+PC)^{(N-1-k)}(PC)^k, \tag{23}$$

the numerator of $\widetilde{S}_{ii}$ is

$$(1+PC)^{(N-1)}$$

and the denominator becomes

$$(1+PC)^N.$$

So we can easily prove the equation (21):

$$\widetilde{S}_{ii} = \frac{(1+PC)^{(N-1)}}{(1+PC)^N} = \frac{1}{PC+1}$$

Then, since the outdegree of each node is either 0 or 1, if there is a path connecting i and j then $\mathcal{L}w_{ij}^k = 1$ and there is at most one directed path between any two vertices. So the equation (23) becomes

$$(1+PC)^{(N-1-k)}(PC)^k.$$

Now we can prove the equation (22)

$$\widetilde{S}_{ij} = \frac{(1+PC)^{(N-1-k)}(PC)^k}{(1+PC)^N} = \frac{(PC)^k}{(1+PC)^{(k+1)}}.$$

■

5.2.1 Leader-follower graph

We have seen that relation (9) holds only if every node has outdegree greater than zero. By definition in a leader-follower topology the leader node has $d_o = 0$, leading it to have a sensitivity transfer function where the degree of the numerator is less than the degree of the denominator. Therefore any leader-follower topology has the asymptotic value of any sensitivity function equal to zero for $|PC| \to \infty$.

The diagonal sensitivity transfer function of an agent with outdegree equal to zero, will be in any case always equal to S. This is because since that agent node is not involved in any cycle, $\mathcal{G}$ and $\mathcal{G}_i$ will have exactly the same cycles, and equations (13) and (4) will differ only by a $(1+PC)$ at the denominator.

5.3 Single cycle directed graph

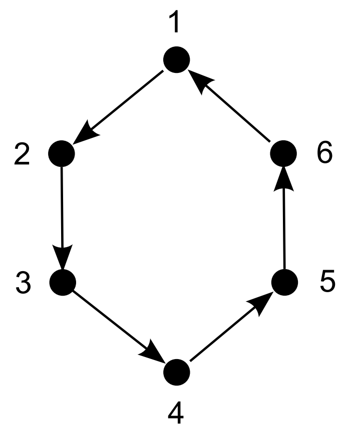


Figure 9: Single directed cycle graph.

Theorem 6. *For a single cycle directed graph with N nodes the networked sensitivity function matrix $\widetilde{S}$ is composed by:*

$$\widetilde{S}_{ii} = \frac{(1+PC)^{(N-1)}}{(1+PC)^N - (PC)^N} \tag{24}$$

and

$$\widetilde{S}_{ij} = \frac{(1+PC)^{(N-1-k)}(PC)^k}{(1+PC)^N - (PC)^N}, \tag{25}$$

where k is the length of the only path from i to j. The asymptotic value when $|PC| \to \infty$ is always equal to $1/N$.

Proof. In this graph there is only a simple cycle of length N and the out degree of each node is equal to one $d_o = 1$. Therefore $\sum_{\mathcal{G}} \left(\mathcal{L}w_{ij}^k\right) = 1$ for all i, j, k, $\sum_{\mathcal{G}} \left(\mathcal{L}w_o^k\right) = -1$, $\sum_{\mathcal{G}_i} \left(\mathcal{L}w_o^{\overline{k}}\right) = \sum_{\mathcal{G}_{ij}^k} \left(\mathcal{L}w_o^{\overline{k}}\right) = 0$ for all i, j because there are no cycles left in any subgraph. Substituting these values in equations (4) (5) and (6) we obtain equations (24) and (25). ■

5.4 Directed star graph

A star graph of order N, sometimes simply known as an N-star, is a tree on N nodes with one node having degree $N-1$ and the other $N-1$ having degree 1 (see an example in Figure 10). We indicate the node of degree $N-1$ as i^*.

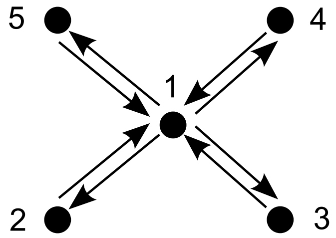


Figure 10: Directed star graph of order 5.

Theorem 7. *In a star graph the diagonal sensitivity function for the node of degree $N-1$, which we write as $\widetilde{S}_{i^*i^*}$, is independent from N and equal to:*

$$\widetilde{S}_{i^*i^*} = \frac{PC+1}{2PC+1}, \tag{26}$$

with an asymptotic value of 0.5. The diagonal sensitivity functions for all the other nodes of degree 1 is

$$\widetilde{S}_{ii} = \frac{\frac{1}{N-1}PC^2 + 2PC + 1}{2PC^2 + 3PC + 1}, \ \forall i \neq i^*, \tag{27}$$

with an asymptotic value of $\dfrac{1}{2(N-1)}$. The off diagonal sensitivity functions are

$$\widetilde{S}_{ii^*} = \frac{PC}{2PC+1}, \tag{28}$$

$$\widetilde{S}_{ij} = \frac{\frac{1}{N-1}PC^2}{2PC^2 + 3PC + 1}, \tag{29}$$

$$\widetilde{S}_{i^*i} = \frac{\frac{1}{N-1}PC}{2PC+1}. \tag{30}$$

Proof. In a directed star graph there are $N-1$ simple cycles of length $k=2$ with Laplacian weight of $\mathcal{L}w_o^k = \frac{-1}{N-1}$ and no disjoint cycles because every cycle passes through the node i^*. In $\mathcal{G}_{i^*}$ and $\mathcal{G}_{ij}^k$ there are no cycles left. So equations (13) and (4) become:

$$(1+PC)^{(N-1)}$$

$$(1+PC)^N - \frac{N-1}{N-1}(1+PC)^{(N-2)}PC^2$$

Then we can prove the equations (26):

$$\widetilde{S}_{i^*i^*} = \frac{(1+PC)^{(N-1)}}{(1+PC)^N - (1+PC)^{(N-2)}PC^2} = \frac{(1+PC)}{(1+PC)^2 - PC^2} = \frac{PC+1}{2PC+1}$$

In $\mathcal{G}_i$ there will be $N-2$ cycles of length $\overline{k}=2$ with Laplacian weight of $\mathcal{L}w_o^{\overline{k}} = \frac{-1}{N-1}$ and no disjoint cycles. So the other diagonal sensitivity functions will be as in the equation (27):

$$\begin{aligned}\widetilde{S}_{ii} &= \frac{(1+PC)^{(N-1)} - \frac{N-2}{N-1}(1+PC)^{(N-3)}PC^2}{(1+PC)^N - (1+PC)^{(N-2)}PC^2} \\ &= \frac{\frac{1}{N-1}PC^2 + 2PC + 1}{2PC^2 + 3PC + 1}\end{aligned}$$

Proofs for equations (28)-(30) come easily considering that:

1. from a generic node $i \neq i^*$ to i^* there is only one path with $k=1$ and $\mathcal{L}w_{ii^*}^k = 1$,
2. from a generic node $i \neq i^*$ to a generic node $j \neq i^*$ there is only one path with $k=2$ and $\mathcal{L}w_{ij}^k = \frac{1}{N-1}$,
3. from i^* to a generic node $i \neq i^*$ there is only one path with $k=1$ and $\mathcal{L}w_{i^*i}^k = \frac{1}{N-1}$.

■

5.5 Generic graph

Consider next the graph and sensitivity functions plotted in Figure 5. In order to construct the Laplacian weight of a path we need to determine the outdegrees of the nodes:

Node	1	2	3	4	5	6
Out degree	1	2	2	1	1	1

Table 1: Out degrees of nodes in graph in Figure 5

Suppose we want to calculate the asymptotic value of $\widetilde{S}_{15}$. From Figure **??** we can see that there are two directed paths from node 1 to node 5: 1-2-3-5 and 1-2-3-4-5. Thus,

$$\mathcal{L}w_{15}^{3} = \frac{1}{d_{o_1}\cdot d_{o_2}\cdot d_{o_3}} = \frac{1}{4},$$
$$\mathcal{L}w_{15}^{4} = \frac{1}{d_{o_1}\cdot d_{o_2}\cdot d_{o_3}\cdot d_{o_4}} = \frac{1}{4}.$$

The subgraph $\mathcal{G}_{15}$ does not contain any cycles and the numerator is

$$\frac{1}{4}(PC)^3(1+PC)^2 + \frac{1}{4}(PC)^4(1+PC).$$

Then we have to apply equation (4) to find the denominator. The Laplacian weights of the cycles in Figure **??** are

$$\mathcal{L}w^{3} = -\frac{1}{d_{o_1}\cdot d_{o_2}\cdot d_{o_6}} = -\frac{1}{2},$$
$$\mathcal{L}w^{5} = -\frac{1}{d_{o_1}\cdot d_{o_2}\cdot d_{o_3}\cdot d_{o_5}\cdot d_{o_6}} = -\frac{1}{4},$$
$$\mathcal{L}w^{6} = -\frac{1}{d_{o_1}\cdot d_{o_2}\cdot d_{o_3}\cdot d_{o_4}\cdot d_{o_5}\cdot d_{o_6}} = -\frac{1}{4},$$

and the denominator is

$$(1+PC)^6 - \frac{1}{2}(1+PC)^3(PC)^3 - \frac{1}{4}(1+PC)(PC)^5 - \frac{1}{4}(PC)^6.$$

We can easily verify equation (9): $1 - 1/2 - 1/4 - 1/4 = 0$. The asymptotic value is given by the coefficients in front of PC^5 and applying equations (12) and (10) we get $2/17 \approx 0.23$, which matches with Figure 5.

6 Design considerations

In the previous sections we have shown how to derive all the sensitivity transfer functions given a topology. Now we will analyze how to design the topology in order to achieve, when possible, desired levels of performance and we will present some design limitations. Furthermore the role of the cycles will be discussed more in detail.

To aid in our designs, we would like to find a relationship between $\widetilde{S}_{ii}$ and $\widetilde{S}_{ij}$. For low loop gains we have

$$\lim_{|PC|\to 0} |\widetilde{S}_{ii}| = 1$$

and

$$\lim_{|PC|\to 0} |\widetilde{S}_{ij}| = 0.$$

For large loop gains, the following theorem provides a partial characterization of the relationship.

Theorem 8. *The networked sensitivity functions satisfy the bounds*

$$|\widetilde{S}_{ji}| \leq |\widetilde{S}_{ii}|, \ \forall i, j \ \forall PC \text{ such that } \left|\frac{PC}{1+PC}\right| < 1 \tag{31}$$

and furthermore

$$|\widetilde{S}_{ji}| \approx |\widetilde{S}_{ii}|, \ |PC| \to \infty \tag{32}$$

Proof. Since the denominator is the same for both, we will focus only on the numerators of $\widetilde{S}_{ji}$ and $\widetilde{S}_{ii}$, which we already know to be the transpose of the cofactors matrix of $(I + \mathcal{L}\widehat{P}\widehat{C})$. Therefore we are interested in finding a relationship between Δ_{ij} and Δ_{ii}. We collect $(1+PC)$ from all terms of the matrix $(I+\mathcal{L}\widehat{P}\widehat{C})$ and we get $(I+\mathcal{L}\widehat{P}\widehat{C}) = (1+PC)F$, where the elements f_{ij} of the matrix F are

$$f_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\dfrac{1}{d_{o_i}}\dfrac{PC}{1+PC} = a_i & \text{if } (i,j) \in G \\ 0 & \text{otherwise.} \end{cases}$$

F is a matrix with diagonal elements equal to 1 and off-diagonal elements smaller than one in absolute value ($|a_i| < 1$). Moreover $\sum_i |f_{ij}| < 1$ for $i \neq j$. The minors of F will be indicated as ΔF_{ii} and ΔF_{ij}. Since $(1 + PC) \geq 0$, the proof will focus on F and it will proceed by induction. The submatrices of F obtained by deleting row i and column i will always have diagonal elements equal to 1 and off-diagonal elements smaller than one in absolute value. The submatrices obtained by deleting row i and column j, with $i \neq j$, will always have diagonal elements equal to 1 except for two that will have absolute value smaller than one. We first prove the theorem for $N = 2$. In this case we can easily see that $\Delta_{ii} = 1$ and $\Delta_{ij} = |a_i|$ or $\Delta_{ij} = 0$, according to the topology, and therefore $\Delta F_{ij} \leq \Delta F_{ii}$. Now we have to show that $\Delta F_{ij} \leq \Delta F_{ii}$ for $N+1$. We start by computing ΔF_{ii} for a matrix of size $(N+1)\times(N+1)$ with the Laplace expansion of the determinant in terms of the minors of the $N \times N$ submatrices:

$$\Delta F_{ii_{N+1}} = \Delta F_{ii_N} + a_i \sum_{j \in \mathcal{J}_i} \Delta F_{ij_N},$$

where $\mathcal{J}_i \subset [1, N]\backslash\{i\}$ is the set of nodes which have an arc with tail in i. To compute $\Delta F_{ij_{N+1}}$ we can always choose a row with all elements less than 1. Therefore we have

$$\Delta F_{ij_{N+1}} = |a_i|\Delta F_{ii_N} + a_i \sum_{j \in \mathcal{J}_i} \Delta F_{ij_N}.$$

Since $|a_i| < 1$, we can conclude that $\Delta F_{ij_{N+1}} \leq \Delta F_{ii_{N+1}}$. By induction it holds for every N.

To prove equation (32), we need to consider the values of $|\widetilde{S}_{ji}|$ and $|\widetilde{S}_{ii}|$ when $|PC| \to \infty$, which are given by the ratio of coefficients of $(PC)^{(N-1)}$ in the complete expression of the numerators of $\widetilde{S}_{ji}$ (12) and $\widetilde{S}_{ii}$ (14) and the common denominator in equation (10). Since the denominator is the same for both, we need to prove that equations (12) and (14) represent the same numerical value. From Proposition 2 it easily follows that equation (12) is the ijth first minor of $\mathcal{L}$, while equation (14) is the iith first minor of $\mathcal{L}$. Since the numerator of $\widetilde{S}_{ji}$ is the transpose of the cofactors matrix, we are interested in the ijth minor of $\mathcal{L}$. Tutte [24] extended the matrix tree theorem by showing that the number of out-trees rooted at node i is the value of any cofactor in the ith row of $\mathcal{L}$. Therefore all the cofactors of the same row of $\mathcal{L}$ have the same value. ■

Looking at inequality (31), we can observe that it holds only for $|PC/(1+PC)| < 1$. This is a condition on the magnitude of the single agent complementary sensitivity function $T = PC/(1+PC)$. Therefore inequality (31) holds when $|T(j\omega)| < 1$. If the control is correctly designed, we have $|T(j\omega)| \approx 1$ for $\omega \ll \omega_c$ and $|T(j\omega)| \ll 1$ for $\omega \gg \omega_c$. The complementary sensitivity function is the closed loop transfer function from the reference signal to the process output but also the transfer function from the measurement noise to the process output. Having $|T(j\omega)| < 1$ means that the process output is reduced with respect to the reference signal and to the noise. Since in a multi-agent system the input of each system depends not only on the own reference signal, but also on the neighbors outputs, the latter can be treated as noise entering the system. If $|T(j\omega)| < 1$ the disturbances affecting the neighbors are not amplified.

The inequality (31) has a physical meaning. It shows that for $|T(j\omega)| < 1$ the disturbance is stronger on the agent on which it acts than on the neighbors and the attenuation is stronger on agents far from the agent on which the disturbance acts. If a disturbance enters on the agent i, all the other agents connected with a path to it are influenced, but less than the agent i itself.

Equation (32) states that for very high gain of the system, the disturbance affecting agent i is propagated with the same intensity through all its neighbors.

6.1 Design limitations

Analyzing the signs of the Laplacian weights we can observe the following: $(\mathcal{L}w_{ij})$ is always positive, for simple cycles $(\mathcal{L}w_o)$ is always negative and for disjoint cycles nothing can be said about $(\mathcal{L}w_o)$ (for example two disjoint cycles of even length are positive, while three disjoint cycles of even length are negative).

As it is defined, the Laplacian weight of a path or a cycle is in modulus always less than or equal to one, $|\mathcal{L}w| \leq 1$. Each Laplacian weight of a disjoint cycle is composed of the Laplacian weights of at least two simple cycles multiplied. Given two simple cycles a and b and the disjoint cycles composed by a and b these relations hold: $|\mathcal{L}w_{o(ab)}| \leq |\mathcal{L}w_{o(a)}|$, $|\mathcal{L}w_{o(ab)}| \leq |\mathcal{L}w_{o(b)}|$, $|\mathcal{L}w_{o(ab)}| \leq |\mathcal{L}w_{o(a)}| + |\mathcal{L}w_{o(b)}|$ and $(\mathcal{L}w_a) + (\mathcal{L}w_b) + (\mathcal{L}w_{ab}) \leq 0$. Therefore:

$$-1 \leq \sum (\mathcal{L}w_o) \leq 0$$

and the more cycles there are in the subgraph, the more negative it is.

Define the global *loopiness* C as the total number of distinct simple cycles in the graph (cyclic permutations of the nodes do not count). The local counterpart, $C^{(i)}$, is the number of simple cycles that pass through node i. Define the loopiness ratio of a node as $C_r^{(i)} = \dfrac{C^{(i)}}{C}$.

The asymptotic value of the sensitivity depends on the loopiness ratio. We have that the lower $C_r^{(i)}$ is, the lower the asymptotic value of $\widetilde{S}_{ii}$ and $\widetilde{S}_{ji}$ will be. This is because we know that $(\mathcal{L}w_o)$ is always negative and therefore we would like to have the highest number of cycles in the subgraph $\mathcal{G}_i$, in order to keep the asymptotic value as low as possible. If a small number of cycles pass through node i compared to the total number of cycles in the graph, in the subgraph $\mathcal{G}_i$ there will be a large number of cycles left and then the asymptotic value will be low.

Unfortunately we cannot bring all the asymptotic values to be small at the same time. If we look at the sum of all the asymptotic values of $\widetilde{S}_{ii}$ for graph with $d_o > 0$, $\forall i \in \mathcal{G}$, we can see that they sum up to the unity:

$$\frac{\sum_{i=1}^{N}\left(1+\sum_{\mathcal{G}_i}(\mathcal{L}w_o)\right)}{N+\sum_{\mathcal{G}}\left(\mathcal{L}w_o^k\right)(N-k)} = \frac{N+\sum_{\mathcal{G}}\left(\mathcal{L}w_o^k\right)(N-k)}{N+\sum_{\mathcal{G}}\left(\mathcal{L}w_o^k\right)(N-k)} = 1$$

This is because a cycle is not counted in $\mathcal{G}_i$ if the node i belongs to the cycle, therefore each cycle in all the $\mathcal{G}_i$ is counted $(N-k)$ times. It implies that there are fundamental limitations to what can be achieved by control and that control design can be viewed as a redistribution of disturbance attenuation at low frequencies among the agents. Thus if we want to keep all the asymptotic values as small as possible, the best result we can achieve is $a = 1/N$ for all the nodes.

We have already seen that this property holds for complete graphs and single cycle directed graphs. But we can extend the class of graphs with this property to the directed regular graphs. A directed regular graph is a graph where each vertex has the same number of neighbors, i.e. every vertex has the same indegree and outdegree. Since in a directed regular graph there is a full symmetry, the diagonal sensitivity functions are the same for all the nodes, leading to the same asymptotic value of $1/N$ as $|PC| \to \infty$. For a generic graph, if we want to keep the asymptotic value as low as possible, we have to equally distribute the cycles on the nodes. The loopiness ratio should be more or less the same for all the nodes.

And what about the worst sensitivity function we can have? We can have the worst case when all the cycles of the graph are concentrated in one node, while every other node has only one cycle, as in a directed star graph.

6.2 Bode's integral formula

Bode [25] showed that for a SISO, stable, open loop system $P(s)C(s)$, the sensitivity function $S(s)$ must satisfy the integral

$$\int_0^\infty \log|S(j\omega)|d\omega = 0.$$

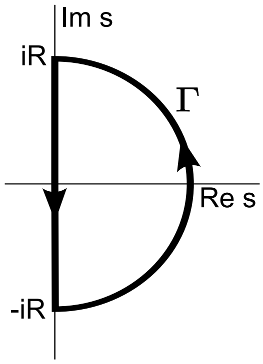


Figure 11: Contour used to prove Bode's theorem.

If the open loop system has unstable poles $\{p_i\}_{i=1}^{n_u}$ then the integral is now equal to

$$\int_0^\infty \log|S(j\omega)|d\omega = \pi \sum_{i=1}^{n_u} \mathrm{Re}(p_i).$$

These integrals imply that if the sensitivity function is made smaller for some frequencies, it must increase at other frequencies so that the integral of $\log|S(j\omega)|$ remains constant. This means that if disturbance attenuation is improved in one frequency range, it will be worse in another, a property sometime referred to as the waterbed effect.

Theorem 9. *In a multi-agent system, Bode's integral formula for stable open loop systems still holds for each diagonal interconnected sensitivity function $\widetilde{S}_{ii}$, no matter what the interconnection topology is.*

Proof. In the single agent case, Bode's formula is derived using the theory of complex variables and contour integration. In the following proof, when possible, the pattern of [26] will be followed. In the multi-agent case we no longer have that the poles of the loop transfer function are the zeros of the sensitivity function. So for simplicity from now on we will consider only stable systems. We assume that the loop transfer function goes to to zero faster than $1/s$ for large values of s. For graphs with at least one node with $d_o = 0$, since all the diagonal sensitivity functions are strictly proper functions, the proof follows the same ideas of the single agent one. We will prove now the theorem for graphs with $d_{oi} > 0$ for all $i \in \mathcal{G}$. Consider the integral of the logarithm of the networked sensitivity function over the contour Γ shown in Figure 11 that encloses the right half plane. The direction of the contour is counterclockwise. The integral of $\log(\widetilde{S}_{ii}(s))$ around this contour is given by

$$\int_\Gamma \log(\widetilde{S}_{ii}(s))ds = \int_{iR}^{-iR} \log(\widetilde{S}_{ii}(s))ds + \int_R \log(\widetilde{S}_{ii}(s))ds = I_1 + I_2 = 0,$$

where R is a large semicircle on the right. The integral is zero because the function $\log(S(s))$ is analytic inside the contour Γ. The first integral is

$$I_1 = -i\int_{-iR}^{iR} \log(\widetilde{S}_{ii}(j\omega))d\omega = -2i\int_0^{iR} \log(|\widetilde{S}_{ii}(j\omega)|)d\omega$$

because the real part of $\log \widetilde{S}_{ii}(i\omega)$ is an even function and the imaginary part is an odd function.

We already know that $\widetilde{S}_{ii}$ is a rational function of the open loop transfer function, therefore we can express it as $\widetilde{S}_{ii}(s) = N(L(s))/D(L(s))$ with $N(L(s))$ and $D(L(s))$ of the same order. The second integral can be rewritten as

$$I_2 = \textstyle\int_R \log(\widetilde{S}_{ii}(s))ds = \int_R \log(N(L(s)))ds - \int_R \log(D(L(s)))ds. \tag{33}$$

Furthermore for $|PC| = 0$, that is for large $|s|$, we have $|\widetilde{S}_{ii}(s)| = 1$, meaning that numerator and denominator have the same constant term that will be called c. Through a Taylor series expansion of the first order we have

$$\begin{aligned} \log(N(L(s))) &\approx \log(c) + \frac{a}{c}L(s), \\ \log(D(L(s))) &\approx \log(c) + \frac{b}{c}L(s), \end{aligned}$$

where a and b are coefficients of $L(s)$ in the polynomials $N(L(s))$ and $D(L(s))$ respectively. Equation (33) becomes

$$I_2 \approx \int_R \log(c)ds + \int_R \frac{a}{c}L(s)ds + - \int_R \log(c)ds - \int_R \frac{b}{c}L(s)ds = \left(\frac{a-b}{c}\right)\int_R L(s)ds.$$

Since $c \neq 0$ and $L(s)$ goes to zero faster than $1/s$ for large s, the integral goes to zero when the radius of the circle goes to infinity. Letting the circle go to infinity gives:

$$\int_0^\infty \log(|\widetilde{S}_{ii}(j\omega)|)d\omega = 0.$$

■

6.3 Limiting cases

Since the best performance can be achieved when the diagonal networked sensitivity functions can be approximated with the single agent one, we now analyze when this approximation holds.

What happens to the networked sensitivity function when N becomes very large? If the cycles are equally distributed on the nodes, as in the single cycle directed graph or in the directed regular graphs, $\widetilde{S}_{ii} \to S$ for $N \to \infty$. It can be easily seen for example in the complete directed graph rewriting equation (16):

$$\lim_{N\to\infty} \widetilde{S}_{ii} = \lim_{N\to\infty} \frac{1 + \dfrac{1}{N-1}PC}{1 + \dfrac{N}{N-1}PC} = \frac{1}{1+PC} = S.$$

The same approximation can be obtained even if the graph is not regular. If the outdegree of each node is very high, $d_o \to \infty$, the Laplacian weights are all very small, $\mathcal{L}w_o \to 0$, and we can get rid of the sum:

$$\lim_{d_o\to\infty} \widetilde{S}_{ii} = \frac{(1+PC)^{(N-1)}}{(1+PC)^N} = \frac{1}{1+PC} = S.$$

There is another way of having small $\mathcal{L}w_o$. The longer all the cycles in the graph are, the higher the possibility of having small Laplacian weights, because of how they are defined, even if the outdegree of each node is not very high.

6.4 Other sensitivity functions

We will now take a look at the other transfer functions. The networked complementary sensitivity $\widetilde{T}(s)$ is the matrix transfer function from any reference signal to all the process outputs and also the matrix transfer function from the measurement noise to the process output. So we would like to have $|\widetilde{T}_{ij}(s)| \approx 1$ in the low-frequency range and $|\widetilde{T}_{ij}(s)| \ll 1$ in the high frequency range where typically the noise concentrates. First of all we have to observe that the single agent constraint $S(s) + T(s) = 1$ does not hold for all the frequencies in a multi-agent system:

$$\widetilde{S}(s) + \widetilde{T}(s) = \left(I + \mathcal{L}_{(n)}\widehat{P}\widehat{C}\right)^{-1}\left(I + \widehat{P}\widehat{C}\right) \neq I$$

The constraint holds only for $|PC| \ll 1$ where $|\widetilde{S}_{ij}(s)| \approx 1$ and $|\widetilde{T}_{ij}(s)| \approx 0$. Since the gain of the loop transfer function is low at high frequencies, the request on the noise rejection is satisfied. In the previous sections we saw that for graphs with $d_{o_i} > 0$ for all $i \in \mathcal{G}$ the networked sensitivity functions are proper transfer functions in PC. Since the networked complementary sensitivity functions can be expressed as $\widetilde{T}_{ij} = PC\widetilde{S}_{ij}$, they are not proper in PC. Usually for $\omega \ll \omega_c$ we have $|PC| \gg 1$ and since $\widetilde{T}_{ij}$ is not proper in PC, it will grow unbounded as the gain of the loop transfer function grows. If we consider instead leader agents and directed tree graphs we have $\widetilde{T}_{ii} = T$ because, like the networked sensitivity functions, the networked complementary sensitivity functions also behave as in the the single agent case for low and high frequencies. Good steady-state command response, expressed by the networked load sensitivity function matrix $\widehat{P}(s)\widetilde{S}(s)$, is strictly related to $\widetilde{S}(s)$ and shows the same performance limits. A different analysis is needed for the networked noise sensitivity function $\mathcal{L}_{(n)}\widehat{C}(s)\widetilde{S}(s)$ because it has $\mathcal{L}_{(n)}$ at the numerator. Every diagonal element is given by

$$\left(\mathcal{L}_{(n)}\widehat{C}(s)\widetilde{S}(s)\right)_{ii} = \sum_{j=1}^{N} Cl_{ij}\widetilde{S}_{ji}.$$

Because of the Laplacian zero row sum property and since $|\widetilde{S}_{ji}| \approx |\widetilde{S}_{ii}|$ for $|PC| \to \infty$, $\lim_{|PC|\to 0} |\widetilde{S}_{ii}| = 1$ and $\lim_{|PC|\to 0} |\widetilde{S}_{ij}| = 0$, we have that at low frequencies $|\mathcal{L}_{(n)}\widehat{C}(s)\widetilde{S}(s)|_{ii} \ll 1$, and at high frequencies $|\mathcal{L}_{(n)}\widehat{C}(s)\widetilde{S}(s)|_{ii} \approx C(s)$.

7 Examples

In this section we apply the theory developed above to some formations and we analyze the frequency domain behavior. Then we present a detailed example on how formation performance requirements translate into control design techniques.

7.1 Five agents with differing topologies

Suppose there are five agents in the formation with identical dynamics $P(s) = 1/(s^2 + s + 4)$ and the local controller $C(s) = (800s+2000)/(s+40)$. It can be shown that $C(s)$ is a stabilizing controller for a single agent with infinite gain margin and 60° of phase margin. For the single agent case the sensitivity function is shown in Figure 12. The magnitude for very low frequencies is $|S| = 7.4 \cdot 10^{-2}$, meaning that each disturbance with low frequency entering the system will be attenuated by a factor of $7.4 \cdot 10^{-2}$.

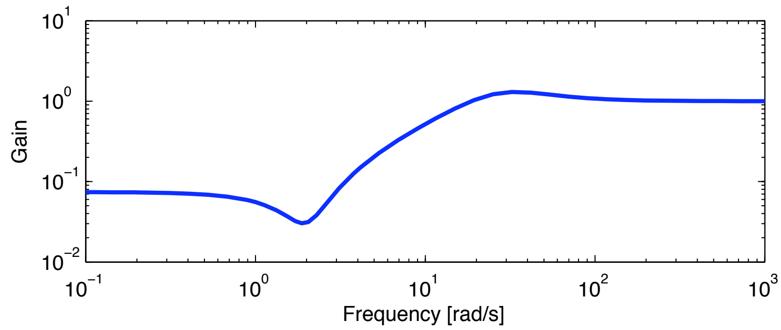
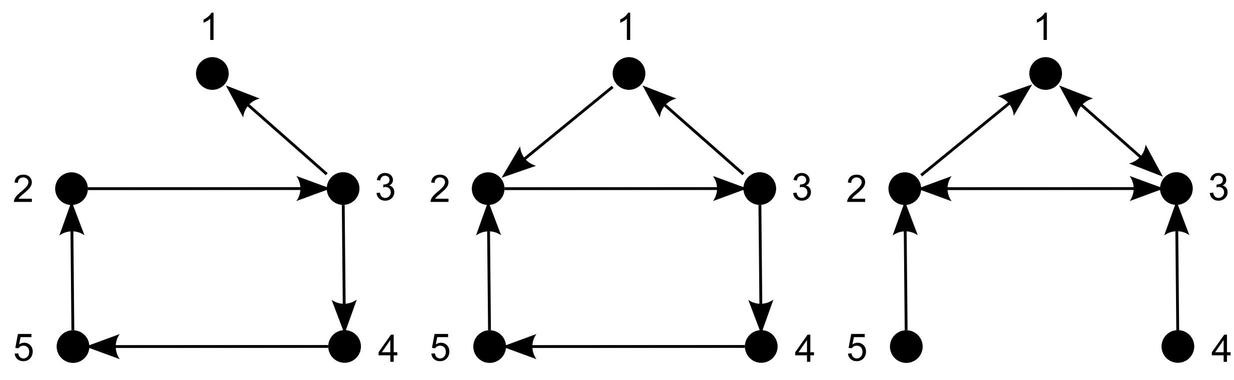


Figure 12: Single agent sensitivity function S.

Let us see now what happens when we have the multi-agent system. In the following examples we will consider only stable topologies and we will focus on performance. Suppose we have the three interaction topologies shown in Figure 13 .



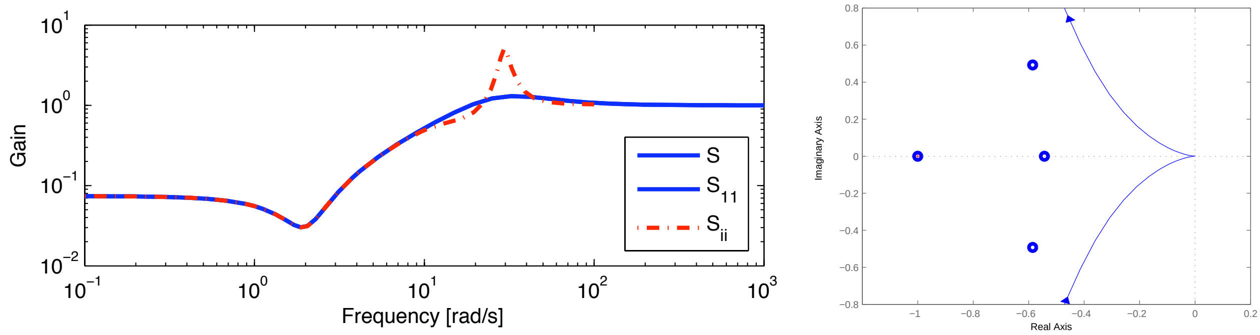
(a) Topology 1 (b) Topology 2 (c) Topology 3

Figure 13: Three different topologies.

Topology 1 represents a leader-follower scheme, where agent 1 is the leader and the others are followers. Using the fact that for low frequencies $|\widetilde{S}_{ji}| \approx |\widetilde{S}_{ii}|$ we write the diagonal sensitivity functions. We already know that $\widetilde{S}_{11} = S$, while the others are all equal to:

$$\widetilde{S}_{ii} = \frac{(1+PC)^4}{(1+PC)^5 - 0.5(1+PC)(PC)^4}, \ \forall i \neq 1$$

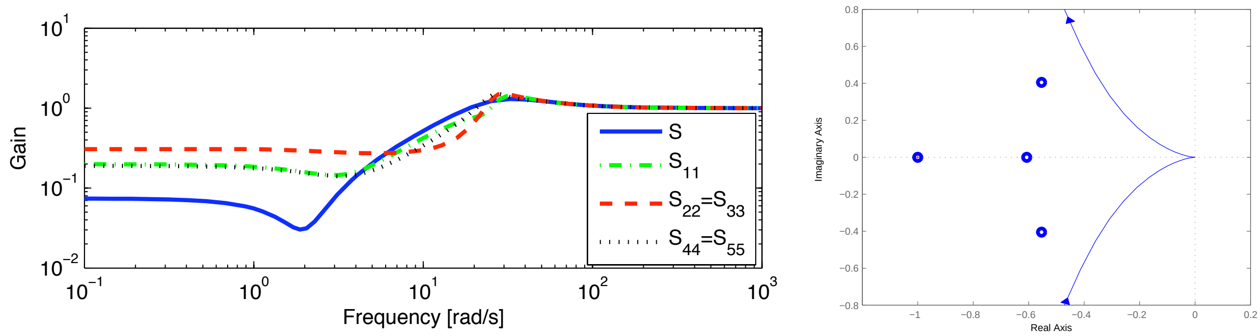
and the Bode plot is shown in Figure 14a. As we expected, for low and high frequencies the leader-follower sensitivity functions behave like the single agent one. But for frequencies near the cut-off frequency, the followers have a high peak value of about $M_s = 5$, meaning that the disturbances on those frequencies will be amplified five times. Even if the system is stable, the interconnection has caused a loss of a significant part of the stability margins, which are now 2.3 for the gain margin and 17.5° for the phase margin (Figure 14b).



(a) Sensitivity functions Bode plot (b) Nyquist plot and critical points

Figure 14: Topology 1.

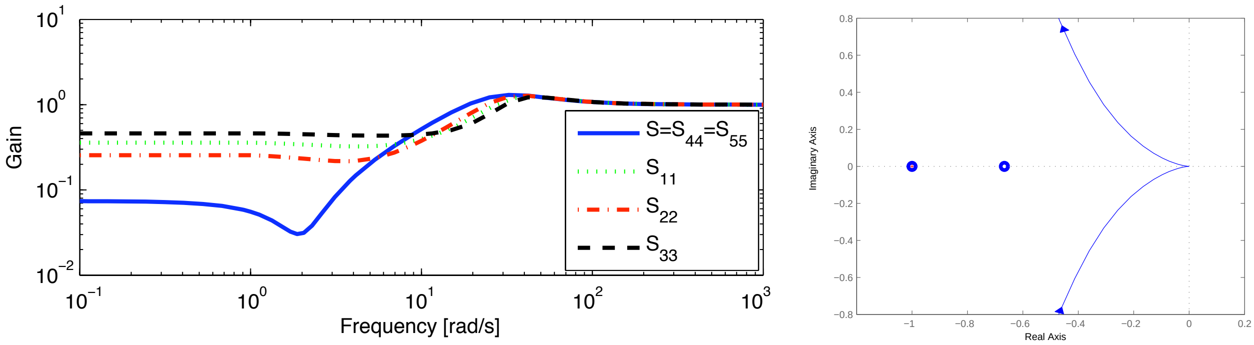
Looking at topology 2 we notice that it is exactly like topology 1 but with arc 12 added. This added arc creates a new cycle and transforms the graph into a strongly connected one. Even if only one arc is added and the critical points in Figure 15b do not move too far from the ones in Figure 14b, the sensitivity transfer functions in Figure 15a are very different from Figure 14a. Comparing Figure 15a with Figure 14a we can see that the disturbance attenuation for low frequencies is worse than in the leader-follower case, having attenuation factors of about 0.3 and 0.2, but near the cut-off frequency the peaks are lower $M_s < 1.5$. As we expected, the poor behavior at low frequencies is given by nodes 2 and 3 because two cycles pass through them while only one cycle passes through nodes 1, 4 and 5. The stability margins for topology 2 are still reduced if compared to the single agent case, but they are better than for topology 1: 2.86 for the gain margin and 19.5° for the phase margin.



(a) Sensitivity functions Bode plot (b) Nyquist plot and critical points

Figure 15: Topology 2.

Consider next topology 3 and look at the Bode plot of the diagonal sensitivity functions in Figure 16a. For low frequencies the disturbance attenuation is clearly worse than the single agent case, leading to have attenuation factors of around 0.47 for node 3, 0.37 for node 1 and 0.26 for node 2. Here there is no rise in the sensitivity function's peak because, even if the graph is directed, the critical points are all real (Figure 16b) and they do not change the stability margins (because the gain margin was and remains infinite).



(a) Sensitivity functions Bode plot (b) Nyquist plot and critical points

Figure 16: Topology 3.

What about the off-diagonal sensitivity functions? As we already know they are always below the corresponding diagonal sensitivity function and they have the same low frequency behavior, while for high frequency their magnitude values decrease. $\widetilde{S}_{ji}$ are band-pass filters and this can be seen in Figure 17, where the Bode plot of some off-diagonal sensitivity functions is shown. The low frequency behavior is similar to the diagonal sensitivity functions, while at high frequencies the gain drastically decreases.

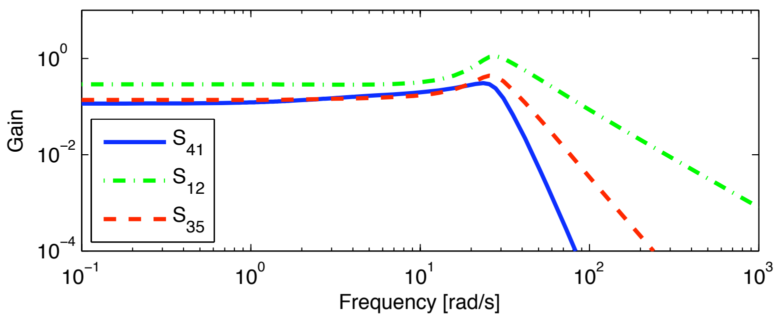


Figure 17: Topology 2: Bode plot of some off-diagonal sensitivity functions.

In Figure 18 Theorem 8 is demonstrated on topology 2. We see in the zoom that $|\widetilde{S}_{12}| > |\widetilde{S}_{22}|$ only for $|T| > 1$.

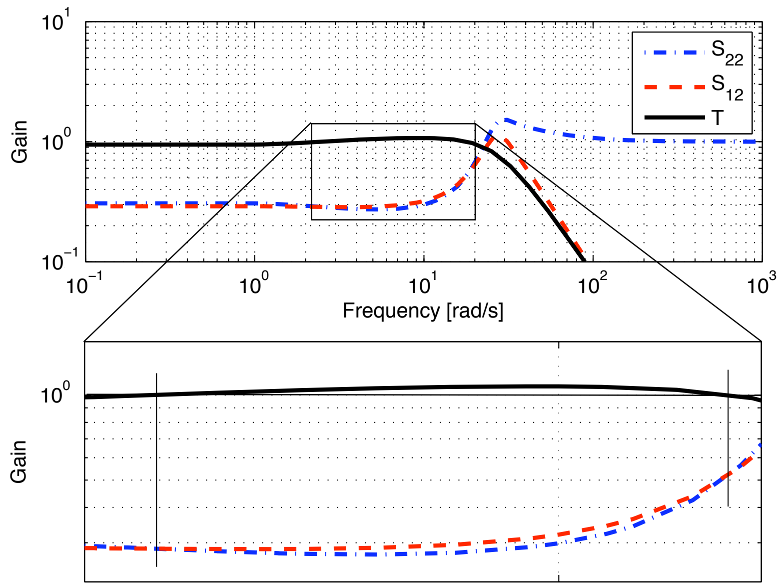


Figure 18: Topology 2: demonstration of Theorem 8.

7.2 Design example

Suppose we have a formation of six vectored thrust aircraft and we want to control the relative roll angle. The dynamics of every agent is modeled with a second-order transfer function as in [26], Example 2.9:

$$P(s) = \frac{r}{Js^2},$$

where $r = 0.25$ m is the force moment arm and $J = 0.0475$ kg m^2 is the vehicle inertia around the roll axis. As local controller we choose the lead compensator of Example 11.6 [26]:

$$C(s) = 200\frac{s+2}{s+50}.$$

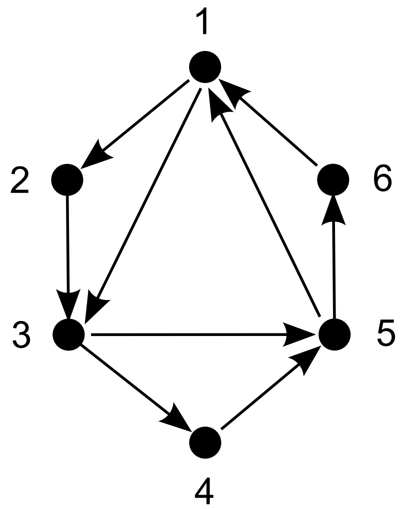
For a single agent system, this controller guarantees less than 1% error in steady state and less than 10% tracking error up to 10 rad/s.

The formation must have no leader and no followers. We would like to design the interconnection topology in order to have low-frequency disturbance rejection less than 30% on agents 1, 3 and 5, less than 10% on agents 2, 4 and 6 and sensitivity high peak M_s less than 1.3 on all the agents.

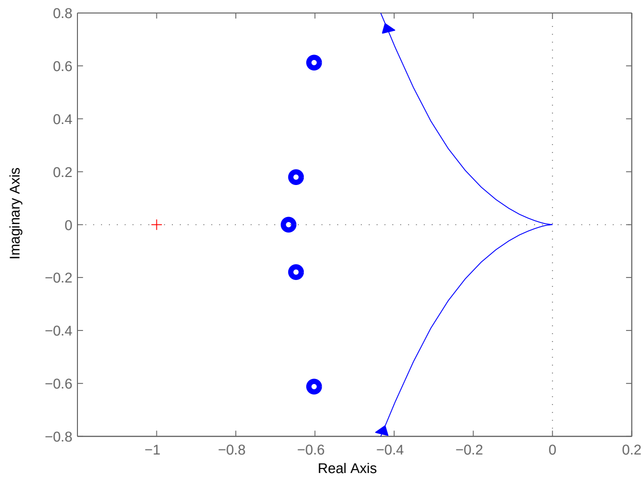
To satisfy the above requirements we need a strongly connected graph. The tighter requirement on agents 2, 4 and 6 translates in designing less cycles passing through these nodes, and more cycles passing through nodes 1, 3 and 5. The sensitivity high peak is related to the Laplacian eigenvalues λ_i, therefore the critical points $-\lambda_i^{-1}$ should be far enough from the $P(s)C(s)$ Nyquist plot. Since we cannot predict the Laplacian spectrum, this aspect will be verified a posteriori.

As a first trial, consider the interconnection topology shown in Figure 19a. The graph is strongly connected and nodes 2, 4 and 6 are part of only one cycle (1-2-3-4-5-6-1), while nodes

1, 3 and 5 are part of two cycles (1-2-3-4-5-6-1 and 1-3-5-1). The multi-agent system is stable (see Figure 19b). Figure 20 shows some of the networked sensitivity functions. The diagonal



(a) Interconnection topology



(b) Nyquist plot and critical points

Figure 19: Aircraft formation, first trial.

sensitivity functions on agents 1, 3 and 5 are the identical. The same holds for agents 2, 4 and 6. At low frequencies we have $|\widetilde{S}_{11}| = 0.21$ and $|\widetilde{S}_{22}| = 0.11$, while the high peaks are $M_s = 1.38$ for $\widetilde{S}_{11}$ and $M_s = 1.32$ for $\widetilde{S}_{22}$. Thus requirements are not satisfied. We need to modify the topology in order to decrease $|\widetilde{S}_{22}|$ at low frequencies and to move the Laplacian eigenvalues.

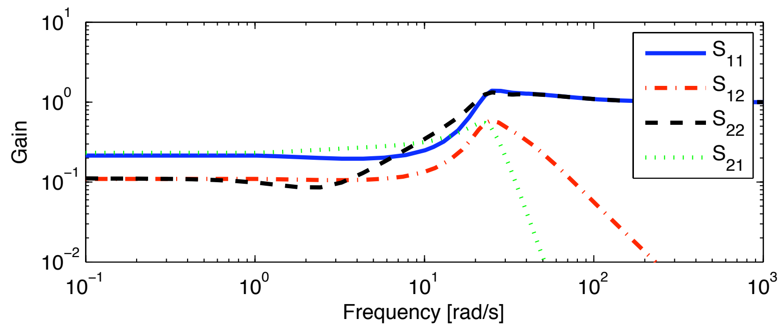
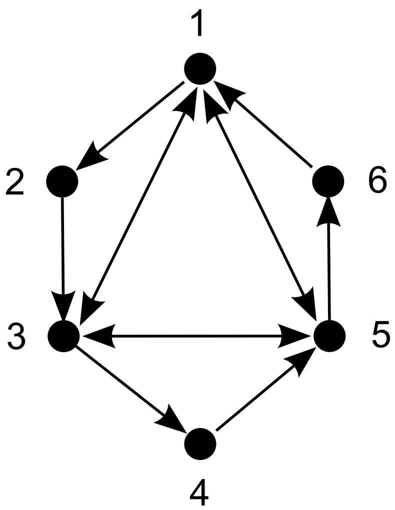
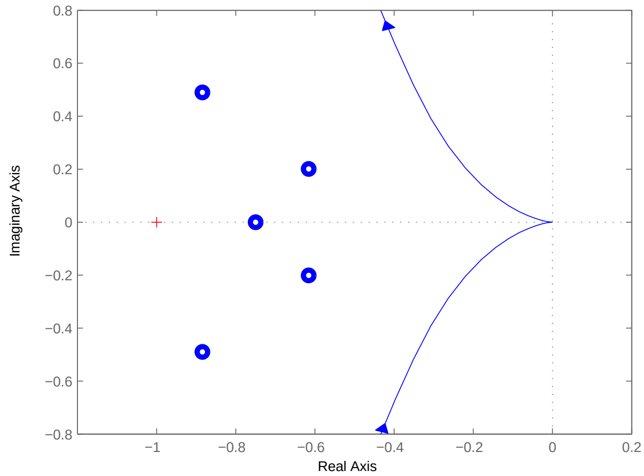


Figure 20: First trial: networked sensitivity functions Bode plot.

As a second trial we add a cycle on nodes 1, 3 and 5, as in Figure 21a. The multi-agent system is still stable and the stability margins are increased (see Figure 21b). We can see in Figure 22 that at low frequencies $|\widetilde{S}_{11}| = 0.24$ and $|\widetilde{S}_{22}| = 0.08$, while the high peaks are $M_s = 1.25$ for $\widetilde{S}_{11}$ and $M_s = 1.27$ for $\widetilde{S}_{22}$. As expected, adding a cycle on nodes 1, 3 and 5 caused a decrease of $|\widetilde{S}_{22}|$ at low frequencies, but also a raise of $|\widetilde{S}_{11}|$ in the same frequency range because of the conservation of the sum of all the asymptotic values of $\widetilde{S}_{ii}$ for $|PC| \to \infty$. The improvement on the stability margins leads to a lower peak of the sensitivity functions.



(a) Interconnection topology



(b) Nyquist plot and critical points

Figure 21: Aircraft formation, second trial.

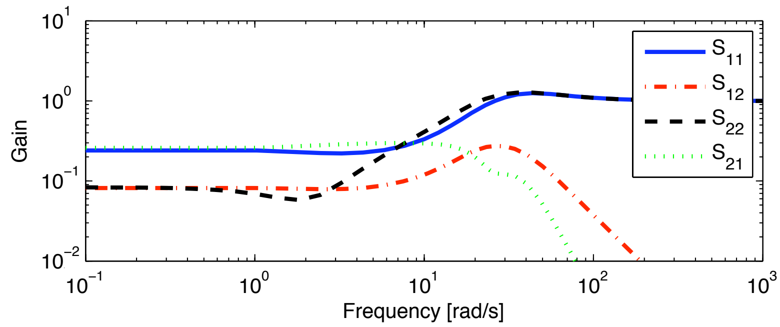


Figure 22: Second trial: networked sensitivity functions Bode plot.

8 Conclusions and future work

In this work, we study how interaction topology influences the behavior of interconnected multi-agent systems in feedback control. We focus on performance analysis and limitations and we suggest design guidelines.

Theorem 1 in Section 4 shows how to compute the network sensitivity functions for arbitrary graphs and number of agents. We expect that this framework, together with the considerations in Section 6, will be helpful for multi-agent controller designers.

From the examples above we can conclude that the interconnection topology influences the sensitivity functions in two ways:

1. the cycles influence the low frequency behavior;
2. the Laplacian spectrum influences the peak value.

Given a topology, the open loop transfer function should have higher gain at low frequencies in order to better attenuate the disturbances. But because of the waterbed effect, a higher gain reduces the stability margins leading to a rise of the sensitivity function's peak.

No matter how the controller is designed, there are fundamental limitations to performance. Control with only feedback does not guarantee disturbance rejection. For this reason, a two degree of freedom controller is needed. In order to improve the properties of the multi-agent system, the feedforward compensation should filter the disturbances arriving from the agent's neighbors.

Our analysis demonstrates that the presence of cycles in the interaction topology degenerates the system's performances. Fax [16] arrived to a similar conclusion when observed that adding a link to a system caused a loss on the stability margin. If there are cycles in the graph, the disturbance entering on an agent passes through its neighbors and comes back making more difficult to attenuate it.

In this paper we have considered only systems with the same identical dynamics, but we expect that this approach can be extended to heterogeneous systems. We conjecture that polynomials of the network sensitivity functions will include different plant models, but that the paths and cycles structures will influence the performances in the same way.

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